ON A SOLUTION OF THE NONLINEAR DIFFUSION EQUATION

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PMM Vol.28, № 4, 1964, pp. 699-707

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(Received April 19, 1964)

Self-oscillatory processes described by nonlinear partial differential equations have been considered in [1 to 7].

Below, a periodic solution to the diffusion equation with a certain nonlinear boundary condition is given.

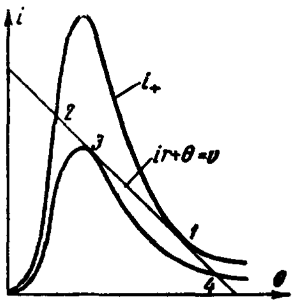

Fig. 1

A set of papers [8 to 11] has been devoted to self-oscillations occuring under definite conditions in electrolytic systems. The theory of this phenomenon is presented briefly in this section and so is the mathematical formulation of the appropriate problem expounded in the mentioned papers.

The theory of self-oscillations [8] detected [9] in the reduction of the $S_2O_8^{2-}$ anions, proceeds from the fact that the system characteristic $P(\theta)$ has a decreasing portion and that the material transport from the solution to the electrode surface is a slow process in which the drop in concentration from the value $c(l,t) = c^\circ$ in the bulk of solution to the value $c(0,t)$ on the surface occurs in a layer of finite thickness l . The current density i and the electrode potential θ are connected [12] by the relationship $i = c(0,t)P(\theta)$ (Fig.1). The system loop consists of two electrodes submerged in an electrolyte and a series resistor R . A constant voltage v is applied to the ends of the loop. The state of the system in the i,θ coordinates is characterized by the intersection (Fig.1) of the line $ir + \theta = v, (r = RS)$, where S is the electrode area, and the curve $i = cP(\theta)$. If there is a stationary state $c(0,t) = c_0$ in the system, then the concentration of material is distributed linearly in the diffusion layer

$$c(x, t) = c_0 + (c^\circ - c_0)\, x/l$$

Under definite conditions the stationary state is unique. If

$$c_0 P' [\theta (c_0)] < -1 / r$$

here, then self-oscillations [11] are excited in the system, and these occur as follows under the assumption that there is no capacitance of the double electric layer: at the initial instant let the system be at the point of tangency 1 (Fig.1) of the line $ir + \theta = v$ to the curve $i_+ = c_+P(\theta)$; it then jumps to the point 2 on the same curve and gradually goes to the point of tangency 3 of the line with the curve $i_- = c_-P(\theta)$ from which it jumps to the point 4 of the same curve, gradually shifts to the point 1 and then the cycle is repeated $(c_- < c_0 < c_+)$. Here $i = nF^*D\partial c(0, t) / \partial x$, where D is the diffusion coefficient , nF^* the consumption of electricity per mole of material.

The mathematical problem reduces to the solution of a nonlinear boundary value problem for the diffusion equation [11]

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < l) \tag{0.1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \frac{1}{Gl}F, \qquad u(l, t) = 0 \qquad (G = nF^*D / l) \tag{0.2}$$

$$u(x, 0) = \Phi(x) \tag{0.3}$$

Here the quantity $u(x,t)$ is related to the concentration $c(x,t)$ as follows:

$$u(x, t) = c(x, t) - c_0 - (c^\circ - c_0)x / l \tag{0.4}$$

$\Phi(x)$ is a certain given function connected with the boundary conditions (in particular $\Phi(l) = 0$); the S-shaped function F (Fig.2) is described by the parametric dependence

$$F = \frac{v - \theta}{r} - G(c^\circ - c_0), \qquad u = -c_0 + \frac{v - \theta}{rP(\theta)} \tag{0.5}$$

The branches $F_1(u)$ and $F_2(u)$ correspond to self-oscillation, where an instantaneous jump from the curve $F = F_1(u)$ to the curve $F = F_2(u)$ is performed at $u = u_+$ and an instantaneous jump from $F = F_2(u)$ to $F = F_1(u)$ at $u = u_-$.

The problem described by Equations (0.1) to (0.3) may be reduced, for a given characteristic $P(\theta)$ to the solution of a nonlinear Volterra integral equation of the second kind whose kernel has a weak singularity [11]. However, in practice it is impossible to determine the polarization curve $P(\theta)$ by experimental means.

A method of finding this curve (for values of θ corresponding to the limit cycle and enclosing the domain of the maximum of the characteristic) by solving the inverse problem is proposed below. The oscillograms $i = i(t)$, obtained experimentally, are here assumed to be known.

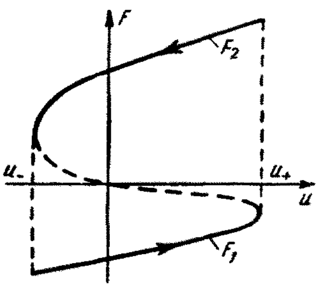

Fig. 2

1. Let us consider the ambiguous function F in condition (0.2) to have the form (Fig.2)

$$F\left[u, \frac{\partial u}{\partial t}\right] = \begin{cases} F_1(u) & \text{for } \partial u/\partial t > 0 \\ F_2(u) & \text{for } \partial u/\partial t < 0 \end{cases} \tag{1.1}$$

$$F_1(u) < 0, \qquad F_2(u) > 0, \qquad F_1'(u_+) = F_2'(u_-) = \infty$$

As is known [13], if the times t proceeding from the beginning of the process are large, the solution of the diffusion equation might be analyzed without the initial conditions.

It has been shown in [10] that if $\lambda = l\sqrt{\pi\nu/D} > 2$ (where ν is the frequency of the process), then the solution of (0.1) might be sought in a semi-infinite domain.

Let us assume that the functions F_1 and F_2 are known functions of the time. Then neglecting the initial condition (0.3), let us seek a periodic solution of the diffusion equation (0.1) in the semi-infinite domain $0 < x < \infty$ with a condition on the $x = 0$ boundary

$$\partial u / \partial x = \chi(t) \tag{1.2}$$

Here $\chi(t)$ is a known periodic function of the time with period T

$$\chi(t) = \begin{cases} \psi(t) & \text{for } \alpha + kT \leqslant t \leqslant \beta + kT \\ \varphi(t) & \text{for } \beta + kT \leqslant t \leqslant \gamma + kT \end{cases} \quad (k = 0, \pm 1, \pm 2, \ldots) \tag{1.3}$$

$$(\alpha = {}^1/{}_2 pT, \quad \beta = -{}^1/{}_2 pT + T, \quad \gamma = {}^1/{}_2 pT + T, \quad 0 < p < 1)$$

the functions $\varphi(t) > 0$ and $\psi(t) < 0$ periodic with period T .

Let us seek the solution of the formulated problem as a Fourier series. Expanding the function $\chi(t)$ in series we find

$$\chi(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\frac{2\pi kt}{T} + b_k \sin\frac{2\pi kt}{T}\right) \tag{1.4}$$

Here

$$a_k = \frac{2}{T}\left[\int_\alpha^\beta \psi(t) \cos\frac{2\pi kt}{T}\,dt + \int_\beta^\gamma \varphi(t)\cos\frac{2\pi kt}{T}\,dt\right] \qquad (k = 0, 1, 2\ldots)$$

$$b_k = \frac{2}{T}\left[\int_\alpha^\beta \psi(t) \sin\frac{2\pi kt}{T}\,dt + \int_\beta^\gamma \varphi(t)\sin\frac{2\pi kt}{T}\,dt\right] \qquad (k = 1, 2, \ldots) \tag{1.5}$$

The expressions for the functions $\partial u(x,t)/\partial x$ and $u(x,t)$ take the form [14]

$$\frac{\partial u(x,t)}{\partial x} = \frac{a_0}{2} + \sum_{k=1}^{\infty} \exp\left\{-\left(\frac{\pi k}{DT}\right)^{1/2} x\right\}\left[a_k \cos\left(\frac{2\pi kt}{T} - \left(\frac{\pi k}{DT}\right)^{1/2} x\right) + \right.$$
$$\left. + b_k \sin\left(\frac{2\pi kt}{T} - \left(\frac{\pi k}{DT}\right)^{1/2} x\right)\right] \tag{1.6}$$

$$u(x,t) = \frac{b_0}{2} + \frac{a_0}{2}x + \frac{1}{2}\left(\frac{DT}{\pi}\right)^{1/2} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \exp\left\{-\left(\frac{\pi k}{DT}\right)^{1/2} x\right\} \times$$
$$\times \left\{(-a_k + b_k)\cos\left[\frac{2\pi kt}{T} - \left(\frac{\pi k}{DT}\right)^{1/2} x\right] - (a_k + b_k)\sin\left[\frac{2\pi kt}{T} - \left(\frac{\pi k}{DT}\right)^{1/2} x\right]\right\} \tag{1.7}$$

By virtue of the second of conditions (0.2), which should now be satisfied as $x \to \infty$, it is necessary to put $a_0 = b_0 = 0$ in (1.6) and (1.7). Hence, the first of Formulas (1.5) yields for $k = 0$

$$\int_{\alpha}^{\beta} \psi(t)\, dt + \int_{\beta}^{\gamma} \varphi(t)\, dt = 0 \tag{1.8}$$

Let us now elucidate which of the conditions should still be imposed on the funcions $\varphi(t)$ and $\psi(t)$ so that condition (1.1) would be satisfied, i.e. so that the solution of the desired nonlinear problem would be obtained.

2. Let us use the Duhamel formula [13]

$$u(x, t) = -\left(\frac{D}{\pi}\right)^{1/2} \int_{-\infty}^{t} \chi(\tau) \exp \frac{-x^2}{4D(t-\tau)} \frac{d\tau}{\sqrt{t-\tau}} \tag{2.1}$$

to describe the solution $u(x, t)$ on each of the two sections $\alpha \leqslant t \leqslant \beta$ ($u_1(x, t)$) and $\beta \leqslant t \leqslant \gamma$ ($u_2(x, t)$).

Putting $x = 0$ into (2.1), we find the value of the function $u(0,t)$ on the boundary

$$u(0, t) = -\left(\frac{D}{\pi}\right)^{1/2} \int_{-\infty}^{t} \chi(\tau) \frac{d\tau}{\sqrt{t-\tau}} \tag{2.2}$$

Replacing the quantity $\chi(\tau)$ in the integrand of (2.2) by the function given by (1.3), and dividing the domain of integration into separate inter vals according to (1.3), where we have put $k = 0, -1, -2, \ldots$, let us represent the functions $u_i(0,t)$ $(i = 1,2)$ as a sign-alternating series

$$u_1(0, t) = -\left(\frac{D}{\pi}\right)^{1/2} \int_{\alpha}^{t} \frac{\psi(\tau)\, d\tau}{\sqrt{t-\tau}} + S_1(t) \tag{2.3}$$

$$u_2(0, t) = -\left(\frac{D}{\pi}\right)^{1/2} \int_{\beta}^{t} \frac{\varphi(\tau)\, d\tau}{\sqrt{t-\tau}} + S_2(t) \tag{2.4}$$

Here

$$S_1(t) = -\left(\frac{D}{\pi}\right)^{1/2} \sum_{k=0}^{-\infty} \left\{ \int_{\beta-T+kT}^{\alpha+kT} \frac{\varphi(\tau)\, dt}{\sqrt{t-\tau}} + \int_{\alpha-T+kT}^{\beta-T+kT} \frac{\psi(\tau)\, d\tau}{\sqrt{t-\tau}} \right\} \tag{2.5}$$

$$S_2(t) = -\left(\frac{D}{\pi}\right)^{1/2} \int_{\alpha}^{\beta} \frac{\psi(\tau)\, d\tau}{\sqrt{t-\tau}} + S_1(t)$$

Using (1.8) which relates the functions $\varphi(t)$ and $\psi(t)$, we easily see that the sufficiency criteria for the convergence of the series $S_1(t)$ and $S_2(t)$ in (2.3) and (2.4) are satisfied.

On the basis of (2.1), (1.2) and (1.3), the expressions for the functions $u_1(x,t)$ and $u_2(x,t)$ may be represented as

$$u_1(x, t) = -\left(\frac{D}{\pi}\right)^{1/2} \int_{\alpha}^{t} \exp \frac{-x^2}{4D(t-\tau)} \frac{\psi(\tau)\, d\tau}{\sqrt{t-\tau}} + S_1(x, t) \tag{2.6}$$

$$u_2(x,t) = -\left(\frac{D}{\pi}\right)^{1/2}\int\limits_{\beta}^{t} \exp\frac{-x^2}{4D(t-\tau)}\frac{\varphi(\tau)\,d\tau}{\sqrt{t-\tau}} + S_2(x,t) \tag{2.7}$$

$$S_1(x,t) = -\left(\frac{D}{\pi}\right)^{1/2}\sum_{k=0}^{-\infty}\Bigg[\int\limits_{\beta-T+kT}^{\alpha+kT} \exp\frac{-x^2}{4D(t-\tau)}\frac{\varphi(\tau)\,d\tau}{\sqrt{t-\tau}} + \int\limits_{\alpha-T+kT}^{\beta-T+kT} \exp\frac{-x^2}{4D(t-\tau)}\frac{\psi(\tau)\,d\tau}{\sqrt{t-\tau}}\Bigg] \tag{2.8}$$

$$S_2(x,t) = -\left(\frac{D}{\pi}\right)^{1/2}\int\limits_{\alpha}^{\beta} \exp\frac{-x^2}{4D(t-\tau)}\frac{\psi(\tau)\,d\tau}{\sqrt{t-\tau}} + S_1(x,t) \tag{2.9}$$

It is easy to show that the sign-alternating series (2.8) and (2.9) converge.

Let us show that it is possible to select the functions $\psi(t)$ and $\varphi(t)$ so that the inequalities

$$\frac{du_1(0,t)}{dt} > 0, \qquad \frac{du_2(0,t)}{dt} < 0 \tag{2.10}$$

would be satisfied.

Let us assume the inequalities (2.10) satisfied.

Then by virtue of (2.3) to (2.5) the minimum and maximum values of the function $u(0,t)$ are given by Formulas (Fig.2)

$$u_- = u_1(\alpha) = u_2(\gamma) = S_1(\alpha) = -\left(\frac{D}{\pi}\right)^{1/2}\int\limits_{\beta}^{\gamma}\frac{\varphi(\tau)\,d\tau}{\sqrt{\gamma-\tau}} + S_2(\gamma)$$

$$u_+ = u_1(\beta) = u_2(\beta) = S_2(\beta) \qquad (u_- < 0,\ u_+ > 0)$$

It follows from (0.2), (1.1) and (1.3) that the functions $F_1(u)$ and $F_2(u)$ will be determined by the parametric relationships

$$F_1 = Gl\psi(t), \qquad u = u_1(0,t) \qquad F_2 = Gl\varphi(t), \quad u = u_2(0,t) \tag{2.11}$$

Hence, we obtain the following formulas for the derivatives:

$$\frac{dF_1}{du} = Gl\frac{\psi'(t)}{u_1'(0,t)}, \qquad \frac{dF_2}{du} = Gl\frac{\varphi'(t)}{u_2'(0,t)} \tag{2.12}$$

If $\psi'(t) > 0$ and $\varphi'(t) < 0$, as they should be from an analysis of the oscillograms [8 and 9], Formulas (2.12) then show that in conformity with Fig.2

$$\frac{dF_i}{du} > 0 \qquad (i = 1, 2)$$

Substituting the functions $F_1(u)$ and $F_2(u)$, defined by (2.11), into (0.5) for intervals of variation θ corresponding to the limit cycle (Fig.1), we obtain dependences $P_1(\theta)$ for $\theta_1 \leqslant \theta \leqslant \theta_4$ and $P_2(\theta)$ for $\theta_2 \leqslant \theta \leqslant \theta_3$. Here, as is seen from Fig.1, the requirements

$$P_1'(\theta) < 0, \qquad \theta_1 \leqslant \theta \leqslant \theta_4 \tag{2.13}$$

$$P_2'(\theta) > 0, \qquad \theta_2 \leqslant \theta < \theta_*, \qquad P_2'(\theta_*) = 0$$

$$P_2'(\theta) < 0, \qquad \theta_* < \theta \leqslant \theta_3 \tag{2.14}$$

should be satisfied.

It is easy to see that if the graph of the function $F_1(u)$ has no inflection points, then to satisfy (2.13) it is sufficient that

$$(c_0 + u_-) F_1'(u_-) \geqslant F_1(u_-) + G(c^\circ - c_0) \tag{2.15}$$

If the graph of the function $F_2(u)$ has no more than one inflection point, then inequalities (2.14) will be satisfied if

$$(c_0 + u_+) F_2'(u_+) < F_2(u_+) + G(c^\circ - c_0) \tag{2.16}$$

3. Let us now turn to condition (2.10). Since

$$\frac{d}{dt}\int_\alpha^t \frac{\psi(\tau)\,d\tau}{\sqrt{t-\tau}} = \frac{\psi(\alpha)}{\sqrt{t-\alpha}} + \int_\alpha^t \frac{\psi'(\tau)\,d\tau}{\sqrt{t-\tau}} \tag{3.1}$$

then (2.3) to (2.5) yield

$$\frac{du_1(0,t)}{dt} = -\left(\frac{D}{\pi}\right)^{1/2}\left[\frac{\psi(\alpha)}{\sqrt{t-\alpha}} + \int_\alpha^t \frac{\psi'(\tau)\,d\tau}{\sqrt{t-\tau}}\right] + \frac{dS_1(t)}{dt} \tag{3.2}$$

$$\frac{du_2(0,t)}{dt} = -\left(\frac{D}{\pi}\right)^{1/2}\left[\frac{\varphi(\beta)}{\sqrt{t-\beta}} + \int_\beta^t \frac{\varphi'(\tau)\,d\tau}{\sqrt{t-\tau}}\right] + \frac{dS_2(t)}{dt} \tag{3.3}$$

where by virtue of (2.5)

$$\frac{dS_1(t)}{dt} = \frac{1}{2}\left(\frac{D}{\pi}\right)^{1/2}\sum_{k=0}^{-\infty}\left\{\int_{\beta-T+kT}^{\alpha+kT} \frac{\varphi(\tau)\,d\tau}{(t-\tau)^{3/2}} + \int_{\alpha-T+kT}^{\beta-T+kT} \frac{\psi(\tau)\,d\tau}{(t-\tau)^{3/2}}\right\} \tag{3.4}$$

$$\frac{dS_2(t)}{dt} = \frac{1}{2}\left(\frac{D}{\pi}\right)^{1/2}\int_\alpha^\beta \frac{\psi(\tau)\,d\tau}{(t-\tau)^{3/2}} + \frac{dS_1(t)}{dt} \tag{3.5}$$

Using (1.8), it is easy to see that the sign-alternating series (3.4) and (3.5) converge for $\alpha < t \leqslant \beta$ and $\beta < t \leqslant \gamma$, respectively, where the sum of the series (3.4) is positive and of (3.5) is negative. Hence, it follows that the inequalities (2.10) will be satisfied if the conditions

$$\frac{\psi(\alpha)}{\sqrt{t-\alpha}} + \int_\alpha^t \frac{\psi'(\tau)\,d\tau}{\sqrt{t-\tau}} < 0 \qquad (\alpha \leqslant t \leqslant \beta) \tag{3.6}$$

$$\frac{\varphi(\beta)}{\sqrt{t-\beta}} + \int_\beta^t \frac{\varphi'(\tau)\,d\tau}{\sqrt{t-\tau}} > 0 \qquad (\beta \leqslant t \leqslant \gamma) \tag{3.7}$$

are imposed on the functions $\psi(t)$ and $\varphi(t)$.

Let us examine the inequality (3.6).

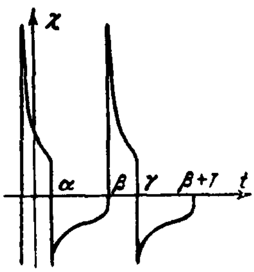

Fig. 3

For t close to α it is satisfied. Let the integral in (3.6) be improper. Since $\psi'(t) > 0$, it cannot be divergent. Therefore, in the neighborhood of the point $t = \beta$ the function $\psi(t)$ should be in the simplest case

$$\psi(t) = e + d(\beta - t)^q \tag{3.8}$$

$$(1/2 < q < 1,\ e < 0,\ d < 0)$$

and the condition $F_1'(u_+) = \infty$ (Fig.2) is satisfied by virtue of (3.8) and (2.12).

Analogously, considering inequality (3.7), we obtain that near the point $t = \gamma$ the function $\varphi(t)$ is described in the simplest case by

$$\varphi(t) = a + b(\gamma - t)^s \qquad (1/2 < s < 1,\ a > 0,\ b > 0) \tag{3.9}$$

4. As an example let us consider the functions $\varphi(t)$ and $\psi(t)$ of the form

$$\varphi(t) = a + b(\gamma - t)^s + g(t - \beta)^{1/2}, \qquad \psi(t) = e + d(\beta - t)^q + f(t - \alpha)^{1/2} \tag{4.1}$$

Here $g < 0$, $f > 0$ and the remaining constants are bounded by the same inequalities as in (3.8) and (3.9). Presented in Fig.3 is the form of the functions $\chi(t)$ given by (4.1).

If the inequalities

$$a + g\sqrt{pT} > 0, \qquad e + f\sqrt{(1-p)T} < 0 \tag{4.2}$$

are satisfied, then here $\varphi(t) > 0$ and $\psi(t) < 0$; as is easy to see by considering (4.1), the derivatives are $\varphi'(t) < 0$ and $\psi'(t) > 0$ Conditions (3.6) and (3.7) are satisfied if

$$e - \frac{d}{2q-1}[(1-p)T]^q + \frac{\pi}{2} f\sqrt{(1-p)T} < 0 \tag{4.3}$$
$$a - \frac{b}{2s-1}(pT)^s + \frac{\pi}{2} g\sqrt{pT} > 0$$

Finally, (1.8) shows that the constants in (3.1) are related by means of the equality

$$apT + \frac{b}{s+1}(pT)^{s+1} + \frac{2}{3} g(pT)^{3/2} + e(1-p)T + \frac{d}{q+1}[(1-p)T]^{q+1} + \\ + \frac{2}{3} f[(1-p)T]^{3/2} = 0 \tag{4.4}$$

Formulas (3.2) and (3.3) show that for t close to α and t close to β, the asymptoyic expansions

$$\frac{du_1(0,t)}{dt} = -\left(\frac{D}{\pi}\right)^{1/2} \frac{[\psi(\alpha) - \varphi(\alpha)]}{\sqrt{t-\alpha}} + \ldots, \\ \frac{du_2(0,t)}{dt} = -\left(\frac{D}{\pi}\right)^{1/2} \frac{[\varphi(\beta) - \psi(\beta)]}{\sqrt{t-\beta}} + \ldots, \tag{4.5}$$

hold, respectively. Returning to Formulas (2.12) and (4.5), we find

$$\frac{dF_1}{du} = -\frac{Glf}{2}\left(\frac{\pi}{D}\right)^{1/2}[\psi(\alpha) - \varphi(\alpha)]^{-1} \quad \text{for } u = u_-$$

$$\frac{dF_2}{du} = -\frac{Glg}{2}\left(\frac{\pi}{D}\right)^{1/2}[\varphi(\beta) - \psi(\beta)]^{-1} \quad \text{for } u = u_+$$

It is clear from (3.2) and (3.3) that for values of t close to β the derivative $du_1(0,t)/dt$ is finite; for t close to γ the derivative $du_2(0,t)/dt$ is finite. Let us put

$$u = u_+ + C_1(t-\beta) \quad (C_1 > 0), \qquad u = u_- - C_2(t-\gamma) \quad (C_2 > 0) \tag{4.6}$$

Then using (2.12), we find in the neighborhood of the value $u = u_+$

$$\frac{dF_1}{du} = -\frac{Gldq}{C_1}\left(\frac{u_+ - u}{C_1}\right)^{q-1}$$

and in the neighborhood of the value $u = u_-$

$$\frac{dF_2}{du} = \frac{Glbs}{C_2}\left(\frac{u - u_-}{C_2}\right)^{s-1}$$

Given in Fig.4 is the form of the function $u(0,t)$ obtained as a result of solving the problem under consideration. It is clear that the oscillations are of the relaxation type.

The inequalities (2.15) and (2.16) will be satisfied in this case if the stronger inequalities

$$f\Big[c_0 + 2\left(\frac{DT}{\pi}\right)^{1/2}\{[a + b(pT)^s](-\sqrt{1+p} + 1 - \sqrt{p}) - (e + f\sqrt{(1-p)T})(1-\sqrt{p})\}\Big] >$$
$$> -2\left(\frac{D}{\pi}\right)^{1/2}\left\{e + d[(1-p)T]^q + \frac{c^\circ - c_0}{l}\right\}\{-a - g\sqrt{pT} + e + d[(1-p)T]^q\} \tag{4.7}$$

$$-g\Big[c_0 + 2\left(\frac{DT}{\pi}\right)^{1/2}\{(-e - d[(1-p)T]^q)(\sqrt{2-p} - 1 + \sqrt{1-p}) + (-a - g\sqrt{pT})(1-\sqrt{1-p})\}\Big] < 2\left(\frac{D}{\pi}\right)^{1/2}\Big[a + b(pT)^s + \frac{c^\circ - c_0}{l}\Big][a + b(pT)^s - e - f\sqrt{(1-p)T}] \tag{4.8}$$

hold, respectively.

Let us note that if we put $g = f = 0$ in (4.1), thereby reducing these expressions to (3.9) and (3.8), respectively, then we will have $F_1'(u_-) = F_2'(u_+) = 0$ at the points of the S-shaped function (Fig.2) characterizing the position of the system after the jumps and thi does not agree with experiment.

5. Let us now assume that near the points $u = u_+$ and $u = u_-$ the functions F_1 and F_2 have the respective forms

$$F_1 = A + B\sqrt{u_+ - u},$$
$$F_2 = C + E\sqrt{u - u_-}$$
$$(A < 0, \quad B < 0, \quad C > 0, \quad E > 0)$$

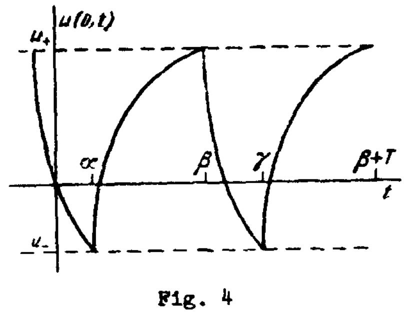

Fig. 4

It is easy to see that the theory developed in the preceeding sections is not applicable for this case. Returning to (2.12), we see that if $u_1'(0,\beta)=0$ at $t=\beta$ and $\psi'(\beta)$ is finite, then the condition $F_1'(u_+)=\infty$ is satisfied; analogously, for $t=\gamma$ we must have $u_2'(0,\gamma)=0$ and $\varphi'(\gamma)$ is finite. For t close to α and t close to β Formulas (4.5), respectively, should hold.

Therefore, we arrive at the following problem: To seek a periodic function $u(x,t)$ in the semi-infinite domain $0<x<\infty$ with the known boundary value

$$u(0, t) = u_0(t) \tag{5.1}$$

Here $u_0(t)$ is a function such as is shown in Fig.4 but with zero derivative at the values

$$t = \alpha + kT - 0, \quad t = \beta + kT - 0$$

The solution of (0.1) with condition (5.1) and without an initial condition is described by the Duhamel formula

$$u(x, t) = \frac{2}{\sqrt{\pi}} \int_0^\infty u_0\left(t - \frac{x}{4D\xi}\right) \exp(-\xi^2)\, d\xi \tag{5.2}$$

Differentiating (5.2) with respect to x and putting $x=0$ in the obtained relationship, we find an expression for the function

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -\frac{1}{(\pi D)^{1/2}} \int_{-\infty}^{t} \frac{u_0'(\tau)\, d\tau}{\sqrt{t-\tau}} \tag{5.3}$$

Now, as in Section 1, let

$$u_0(t) = \begin{cases} v(t), & \alpha + kT \leqslant t \leqslant \beta + kT \\ w(t), & \beta + kT \leqslant t \leqslant \gamma + kT \end{cases} \quad (k = 0, \pm 1, \pm 2, \ldots) \tag{5.4}$$

Then (5.3), analogously to (2.2) to (2.5), will yield

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} = -\frac{1}{(\pi D)^{1/2}} \int_\alpha^t \frac{v'(\tau)\, d\tau}{\sqrt{t-\tau}} + \Phi_1(t) \tag{5.5}$$

$$\left.\frac{\partial u_2}{\partial x}\right|_{x=0} = -\frac{1}{(\pi D)^{1/2}} \int_\beta^t \frac{w'(\tau)\, d\tau}{\sqrt{t-\tau}} + \Phi_2(t) \tag{5.6}$$

in which

$$\Phi_1(t) = -\frac{1}{(\pi D)^{1/2}} \sum_{k=0}^{-\infty} \left\{ \int_{\beta-T+kT}^{\alpha+kT} \frac{w'(\tau)\, d\tau}{\sqrt{t-\tau}} + \int_{\alpha-T+kT}^{\beta-T+kT} \frac{v'(\tau)\, d\tau}{\sqrt{t-\tau}} \right\} \tag{5.7}$$

$$\Phi_2(t) = -\frac{1}{(\pi D)^{1/2}} \int_\alpha^\beta \frac{v'(\tau)\, d\tau}{\sqrt{t-\tau}} + \Phi_1(t)$$

It is seen from (5.5) to (5.7) that a condition analogous to (1.8) is satisfied automatically so that the sign-alternating series (5.5) to (5.7) converge.

In order for the inverse problem given by (5.5) to (5.7) to describe a self-oscillatory process of the considered kind, it is sufficient that the inequalities

$$-\frac{\partial u_1}{\partial x}\Big|_{x=0} < 0, \quad \frac{\partial u_2}{\partial x}\Big|_{x=0} > 0, \quad \frac{d}{dt}\left(\frac{\partial u_1}{\partial x}\Big|_{x=0}\right) > 0, \quad \frac{d}{dt}\left(\frac{\partial u_2}{\partial x}\Big|_{x=0}\right) < 0 \quad (5.8)$$

and also (if the assumptions noted in Section 2 hold) the inequalities (2.15) and (2.16) be satisfied.

In conclusion, I am grateful to A.Ia.Gokhshtein for a number of valuable comments.

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Translated by M.D.F.